Beauty of Mathematics Decal PSET #1 Solutions

1. A very special island is inhabited only by knights and knaves. Knights always tell the truth, knaves always lie.

   You meet six inhabitants: Betty, Rex, Peggy, Carl, Zeke and Marge.
   Betty claims that Marge is a knave.
   Rex tells you that Zeke could say that Peggy is a knight.
   Peggy tells you, “Only a knave would say that Marge is a knave.”
   Carl tells you that either Marge is a knight or Peggy is a knight.
   Zeke tells you, “Neither Marge nor Rex are knights.”
   Marge claims that it's not the case that Rex is a knave.

   Who are the knights and who are the knaves?

   Betty and Zeke are the knights. Marge, Rex, Peggy, and Carl are the knaves.

   Begin by assuming that Betty is a knight. Then Marge is a knave. Which means that Rex is also a knave. Since Rex is lying, Zeke could not say that Peggy is a knight. We know that Marge is a knave, So Peggy has to be lying. Which also means that since Zeke could not say that Peggy is a knight, Zeke is a knight. Last left is Carl, who claims that Marge or Peggy are knights. Neither of them are, so Carl is a knave. And thus we have a consistent system.

   The easiest way to work through a problem like this is to pick a simple statement, assume it is either true or false, and then follow what that statement implies. It is almost like a proof by contradiction- if we had assumed Betty were a knave, we would have eventually arrived upon someone being both a knight and a knave (which is our contradiction).

2. Prove the following statement:

   “For every integer x, if x is even, then for every integer y, xy is even.”

   Proof:
   An even number is one that can be divided by 2 with no remainder. So x can be written as x=2a for a specific a. Examine xy for any particular y. Substitute x with 2a to get xy=2ay. Since 2ay can be divided by 2 with no remainder, xy is even. QED.

(There are other ways you could do this one! You could try proving it by contradiction "Assume x is even and for any integer y, xy is not even." or by contraposition "if xy is not even for every integer y, then x cannot be even")

3. (Tougher!) Explain what is wrong with the following argument:

$0 = 0$
$\therefore\ 0 = 0 + 0 + 0 + 0 + \ldots.$
$\therefore\ 0 = (1-1) + (1-1) + (1-1)+ \ldots.$
$\therefore\ 0 = 1 - 1 + 1 - 1 + 1 - 1 + \ldots.$
$\therefore\ 0 = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots$
$\therefore\ 0 = 1 + 0 + 0 + 0 + \ldots$
$\therefore\ 0 = 1.$

(The "∴" means "therefore" and "..." means continue infinitely many times.)

The reason the argument is faulty is the jump from the line

$\therefore\ 0 = 1 - 1 + 1 - 1 + 1 - 1 + \ldots.$

to

$\therefore\ 0 = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots$

A lot of student solutions said this was impossible because there weren't enough 1's at the end of the equation. This is incorrect! As indicated at the end of the problem, the "..." means to continue that operation infinitely many times. So we have an infinite list 1+1-1+1-1+1-1+1…. Etc. going on forever. We will never run out of 1's!

So why is it that we can't rearrange parentheses like this? This property is called *associativity* and while we're very familiar with using it for finite equations, like

$3 + 4 + 5 = (3+4) + 5 = 3 + (4+5) = 12,$

This doesn't hold for infinite calculations!

In fact, for the infinite sum 1 - 2 + 3 - 4 + 5 - 6 … (continue for every number) if you rearrange the terms and group them together, you can make this sum add up to *anything*, even -1/12 or 65! And clearly those two numbers don't equal each other.

So for *infinity*, we're not allowed to group things together or moving things around without messing up the calculation. It's a strange property, but when you start thinking about infinities, (and different sizes of them), there are entirely new sets of mathematical rules you have to abide by!

Interestingly enough, there is a new research being done on the kind of infinity that you get when you change around numbers in infinite sums. And that type of infinity is possibly *in between* countable and uncountable, which Sander talked about during our third lecture. Ask about it in office hours or email us if you're curious for more!